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Quantum Entropy of Spin Fields in the Schwarzschild-Anti-de Sitter Black Hole with a Global Monopole

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The quantum entropies of gravitational, electromagnetic, neutrino and scalar fields in the *static* Schwarzschild-anti-de Sitter black hole with a global monopole are investigated by using the brick-wall model. The quantum entropy contain two parts: One is quadratically divergent term which takes a geometric character; the other is spin-dependent, logarithmically divergent terms. The whole expression of the entropy of a spin field does not take the form of the scalar field.

KEY WORDS: quantum entropy, spin fields, Schwarzschild-anti-de Sitter black hole with a global monopole, brick-wall model

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1. INTRODUCTION

One of the attractive problems is the study of black hole thermodynamics in the theoretical physics. It is a remarkable and long-known fact that the classical Bekenstein–Hawking entropy of a four-dimensional black hole is proportional to the area of the horizon, i.e., $S = A_h/4$, where A_h is the area of the black hole horizon. *However, the microphysical explanation of the entropy as a counting of states is still absent, though many attempts have been made. One possible way is to associate the entropy with a thermal bath of fields propagating just outside the horizon.*[(1)] *The bath could cause the change of the space-time geometry by back reaction.* It is reasonable to ask whether this geometric character of the entropy remain valid when quantum fluctuations of the matter field in the black

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hole background[1] are taken into account. Much attention has been focused on this problem.[1–19] Using the brick-wall model,[1] 't Hooft found the quantum entropy of the Schwarzschild black hole due to the scalar takes the geometric character $A_h/(48\pi\epsilon^2)$ (which reduces to the Bekenstein–Hawking result when the cutoff satisfies a certain condition), where ϵ is the infrared cutoff. However, using the Euclidean path integral method, Solodukhin[2] found that in addition to the 't Hooft result, the entropy still has a logarithmically divergent term $S_{\ln} = (1/45)\ln(\Gamma/\epsilon)$, where Γ is the infrared cutoff. It has been shown[8–19] that the Bekenstein-Hawking result is only the leading contribution in the integral of an entropy density over a cross section of the horizon, i.e., $S = \oint_H \rho_s d^2x$, because $\rho_s = 1/4$ in the Bekenstein–Hawking entropy.[20]

It is widely believed that the symmetries which exist in the early universe will break spontaneously as the universe expands and cools. Such cosmological phase transitions can give rise to various kinds of topological defects. *The global monopole is the most important one of these defects. As a result from a global symmetry breaking, the global monopole was introduced by Barriola and Vilenkin in 1989.*[21] *The global monopole has Goldstone field with energy density decreasing with the distance only as r^{-2}, so that the total energy is linearly divergent at large distances. The large energy in the Goldstone field surrounding global monopole suggests that they can produce strong gravitational field. The appearance of the global monopoles may have important cosmological consequences: they can produce observational effects and may also affect the galaxy formation.*[21] Recently, using different forms of the metric, Zhang *et al.*[22] and Han *et al.*[23] studied some properties of the global monopole in the anti-de Sitter space-time separately. In this paper, we give the calculation of the quantum entropy of arbitrary spin fields in the *static* Schwarzschild-anti-de Sitter black hole with a global monopole and investigate the contribution of the particles of the spin field to the entropy.

2. SPACE-TIME

The metric of the *static* Schwarzschild-anti-de Sitter black hole with a global monopole can written as[23]

$$ds^2 = Bdt^2 - B^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2, \tag{1}$$

with

$$B = 1 - \eta^2 - \frac{2M}{r} - \frac{\Lambda r^2}{3}, \tag{2}$$

where M is the mass of the black hole, $\Lambda < 0$ is cosmological constant, η is the energy scale of symmetry breaking. *The metric (1) describes a space with a deficit solid angle.* The event horizons are determined by $B = 0$. So long as

$0 > 1 - \eta^2 > \sqrt[3]{9M^2\Lambda}$, there exist three real roots, which are given by

$$\begin{aligned} r_{-1} &= \sqrt{2(1-\eta^2)^3/\Lambda^3}\cos(\alpha/3 - \pi/3) \\ r_1 &= \sqrt{2(1-\eta^2)^3/\Lambda^3}\cos(\alpha/3 + \pi/3) \\ r_H &= \sqrt{2(1-\eta^2)^3/\Lambda^3}\cos(\alpha/3), \end{aligned} \tag{3}$$

with $\cos\alpha = -3M\sqrt{(1-\eta^2)^3/\Lambda^3}$, $r_H > r_1 > 0$, which are the outer and inter horizon, and $r_{-1} < 0$, which is unphysical. Thus, B can be factorized into

$$B = -\frac{\Lambda}{3r}(r - r_H)(r - r_1)(r - r_{-1}) \tag{4}$$

The surface gravity of the black hole horizon is

$$\kappa = -\frac{\Lambda}{6r_H}(r_H - r_1)(r_H - r_{-1}) \tag{5}$$

In order to express the field equations in space-time (1) in the Newman-Penrose formalism, we take covariant components of the null tetrad vectors as

$$\begin{aligned} &l^\mu = (B^{-1}, 1, 0, 0); \qquad && n^\mu = \frac{1}{2}(1, -B, 0, 0) \\ &m^\mu = \frac{1}{\sqrt{2}r}\left(0, 0, 1, \frac{i}{\sin\theta}\right); \quad && \bar{m}^\mu = \frac{1}{\sqrt{2}r}\left(0, 0, 1, -\frac{i}{\sin\theta}\right) \end{aligned}, \tag{6}$$

Using the Newman–Penrose formula,[24] we can get from (1) and (6) the non-vanishing spin-coefficients and the only non-vanishing component of the Weyl tensor

$$\begin{aligned} &\alpha = -\beta = -\frac{ctg\theta}{2\sqrt{2}r}, \qquad \rho = -\frac{1}{r}, \\ &\mu = -\frac{B}{2r}, \quad \gamma = \frac{B'}{4}, \quad \Psi_2 = -\frac{M}{r^3} - \frac{\eta^2}{6r^2}. \end{aligned} \tag{7}$$

Equations (7) tell us that the metric (1) is of Petrov type D.

3. SPIN FIELDS

Using the result of Teukolsky, the field equations of spin $s = \frac{1}{2}$, 1 and 2 for the source-free case in a type D space-time can be combined into[12]

$$\begin{cases} \{[D - (2p+1)\rho][\Delta - 2p\gamma + \mu] \\ \quad -[\delta + 2(p-1)\alpha][\bar{\delta} - 2p\alpha] - (2s-1)(s-1)\Psi_2\}\Omega_{p=s} = 0, \\ \{[\Delta - 2(p+1)\gamma + (1-2p)\mu][D - \rho] \\ \quad -[\bar{\delta} - 2(p+1)\alpha][\delta + 2p\alpha] - (2s-1)(s-1)\Psi_2\}\Omega_{p=-s} = 0 \end{cases} \tag{8}$$

where D, Δ and δ are the directional derivatives, given by $D = l^\mu \partial_\mu$, $\Delta = n^\mu \partial_\mu$, $\delta = m^\mu \partial_\mu$; the first equation is for spin states $p = s$ and the other one is for $p = -s$.

With the transformations: $\Omega_p = r^{p-s}\, {}_pR_{lE}(r)\, {}_pY_{lm}(\theta, \varphi)\, e^{-iEt}$, Eq. (8) can be split into the forms

$$\begin{aligned}&\left\{(r^2B)^{-p}\left[\frac{d}{dr}(r^2B)^{p+1}\frac{d}{dr}\right] + \frac{r^2E^2 + ipE(2rB - r^2B')}{B}\right.\\ &\quad - \frac{2\Lambda r^2 + \eta^2}{3}\left[(2p+1)(p+1) + (s-2)(s-1)\left(s - \frac{1}{2}\right)\right] \\ &\quad \left. + p + s - \lambda^2\right\}\, {}_pR_{lE}(r) = 0,\end{aligned} \tag{9}$$

$$\begin{aligned}&\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right.\\ &\quad \left. + \frac{2ip\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi} - p^2ctg^2\theta + p + \lambda^2\right]{}_pY_{lm}(\theta, \varphi) = 0.\end{aligned} \tag{10}$$

Equation (10) shows that ${}_pY_{lm}(\theta, \varphi)$ is the spin-weighted spherical harmonic,[12,13] and the separation constant λ satisfies $\lambda^2 = (l - p)(l + p + 1)$, and l, m are integers satisfying the inequalities $l \geq s$ and $-l \leq m \leq l$. Equations (9) and (10) can also be shown to describe the behavior of a mass-less scalar field $(p = s = 0)$.

4. STATISTICAL ENTROPY

For calculation of the free energy we adopt the following WKB approximation. We rewrite the radial functions as

$${}_pR_{lE}(r) = \exp[iS(r, p, l, E,)] \tag{11}$$

From Eq. (9), we obtain the radial wave number $k(r, p, l, E) \equiv \partial_r S(r, p, l, E)$:

$$\begin{aligned}k^2 = \frac{E^2}{B^2} - \frac{1}{r^2B}&\left[\frac{2\Lambda r^2 + \eta^2}{3}((2p+1)(p+1)\right.\\ &\left. + (s-2)(s-1)\left(s - \frac{1}{2}\right)\right) - p - s + (l-p)(l+p+1)\Bigg].\end{aligned} \tag{12}$$

According to the semi-classical quantization rule, the wave number is quantized as

$$\int_{r_H+\varepsilon}^{r_H+\varepsilon+L} k(r, p, l, E)dr = n\pi, \qquad n \in N, \tag{13}$$

under the brick wall boundary conditions: ${}_pR_{lE}(r) = 0$ at $r = r_H + \varepsilon$ and $r \geq r_H + \varepsilon + L$, where ε is the distance of the brick wall from the horizon and it satisfies $0 < \varepsilon << r_H$; L is the thickness of the brick wall and it satisfies $L >> \varepsilon$ (here in order to ensure that the field and black hole are in stably thermal equilibrium, L can not be taken as $L >> r_H$.). Then the number of eigen-states with energy smaller than E reads

$$\begin{aligned} g(E) &= \sum_p \sum_l (2l+1)n \\ &= \frac{1}{\pi} \sum_p \int_s^{l_{\max}} (2l+1)dl \int_{r_H+\varepsilon}^{r_H+\varepsilon+L} \frac{dr}{B} \Bigg\{ E^2 - \frac{B}{r^2} \Bigg[\frac{2\Lambda r^2 + \eta^2}{3} \\ &\quad \times \left((2p+1)(p+1) + (s-2)(s-1)\left(s - \frac{1}{2}\right)\right) \\ &\quad - (p+s) + (l-p)(l+p+1)] \Bigg]\Bigg\}^{1/2}, \\ &= \frac{2}{3\pi} \sum_p \int_{r_H+\varepsilon}^{r_H+\varepsilon+L} \frac{r^2 dr}{B^2} \left[E^2 - \frac{Bf(r,p)}{r^2} \right]^{3/2} \end{aligned} \tag{14}$$

where $l_{\max}$ is determined by Eq. (12),while

$$f(r,p) = \frac{2\Lambda r^2 + \eta^2}{3} \left((2p+1)(p+1) + (s-2)(s-1)\left(s - \frac{1}{2}\right)\right) - 2p. \tag{15}$$

The free energy with an inverse temperature β is given by

$$-\beta F = \pm \sum_\alpha \ln(1 \pm e^{-\beta E_\alpha}), \tag{16}$$

The plus sign in Eq. (16) corresponds to the Fermi case, while the minus sign corresponds to the Bose case. Using Eq. (14) to determine the density of states, we obtain the main contribution of the free energy

$$\begin{aligned} F &= -\int_0^\infty \frac{g(E)dE}{e^{\beta E} \pm 1} = -\frac{2}{3\pi} \sum_p \int_{r_H+\varepsilon}^{r_H+\varepsilon+L} \frac{r^2 dr}{B^2} \\ &\quad \times \int_0^\infty \frac{dE}{e^{\beta E} \pm 1} \left[E^2 - \frac{Bf(r,p)}{r^2} \right]^{3/2} \end{aligned} \tag{17}$$

$$= \begin{cases} -\dfrac{2\omega\pi^3}{45\beta^4}\displaystyle\int_{r_H+\varepsilon}^{r_H+\varepsilon+L}\frac{r^2dr}{B^2} + \frac{\pi}{6\beta^2}\sum_p\int_{r_H+\varepsilon}^{r_H+\varepsilon+L}\frac{f(r,p)\,dr}{B} & \text{(bosons)} \\ -\dfrac{7\omega\pi^3}{180\beta^4}\displaystyle\int_{r_H+\varepsilon}^{r_H+\varepsilon+L}\frac{r^2dr}{B^2} + \frac{\pi}{12\beta^2}\int_{r_H+\varepsilon}^{r_H+\varepsilon+L}\frac{f(r,p)\,dr}{B} & \text{(fermions)} \end{cases},$$

where $\omega = \sum_p 1$, For the gravitational and electromagnetic fields $\omega = 2$; For the neutrino and scalar fields $\omega = 1$. Using the formula $S = \beta^2\partial F/\partial\beta$, Choosing the inverse temperature to correspond to the Hawking temperature of the non-extreme black hole, i.e., $\beta = 2\pi/\kappa$, and noting that the area of the event horizon is given by $A_H = \int d\varphi \int d\theta [\sqrt{g_{\theta\theta}g_{\varphi\varphi}}]_{r=r_H}$, we obtain the following expressions of the statistical entropies

$$S = \begin{cases} \dfrac{\omega A_H}{360\beta\varepsilon} + \omega\left[\dfrac{2\pi r_H}{45\beta} + \dfrac{\Lambda r_H^2}{180}\right]\ln\dfrac{L}{\varepsilon} - \dfrac{s(s+1)\left(2\Lambda r_H^2 + \eta^2\right)}{12}\ln\dfrac{L}{\varepsilon} & \text{(bosons)} \\ \dfrac{7\omega A_H}{2880\beta\varepsilon} + \dfrac{7\omega}{8}\left[\dfrac{2\pi r_H}{45\beta} + \dfrac{\Lambda r_H^2}{180}\right]\ln\dfrac{L}{\varepsilon} & \\ \quad -\dfrac{1}{24}\left[\dfrac{2\Lambda r_H^2 + \eta^2}{3}(2p+1)(p+1) + 2p\right]\ln\dfrac{L}{\varepsilon} & \text{(fermions)}, \end{cases} \tag{18}$$

where $A_H = 4\pi r_H^2$ is the horizon area. If we set $l_p^2 = 2\epsilon^2/15$ and $\Gamma^2 = L\epsilon^2/\varepsilon$, where $l_p = \sqrt{\varepsilon\beta/\pi}$ is the proper distance of the brick wall from the horizon, the statistical-mechanical entropy (18) then becomes

$$S = \begin{cases} \omega\dfrac{A_H}{48\pi\epsilon^2} + \omega\left[\dfrac{4\pi r_H}{45\beta} + \dfrac{\Lambda r_H^2}{90}\right]\ln\dfrac{\Gamma}{\epsilon} - \dfrac{s(s+1)\left(2\Lambda r_H^2 + \eta^2\right)}{6}\ln\dfrac{\Gamma}{\epsilon} & \\ \text{(bosons)} & \\ \dfrac{7\omega}{8}\dfrac{A_H}{48\pi\epsilon^2} + \dfrac{7\omega}{8}\left[\dfrac{4\pi r_H}{45\beta} + \dfrac{\Lambda r_H^2}{90}\right]\ln\dfrac{\Gamma}{\epsilon} & \\ \quad -\frac{1}{12}\left[\frac{2\Lambda r_H^2+\eta^2}{3}(2p+1)(p+1) + 2p\right]\ln\frac{\Gamma}{\epsilon} & \text{(fermions)}. \end{cases} \tag{19}$$

5. DISCUSSION

We have investigated the statistical entropies of gravitational, electromagnetic, neutrino and scalar fields on the background of the *static* Schwarzschild-anti-de Sitter black hole with a global monopole by using the brick-wall model. From Eq. (19), it is clear that the quantum entropy of any spin field contains two parts: The first part is quadratically divergent on the horizon, which is an intrinsic contribution from the horizon and takes a geometric character; the second part is logarithmically divergent, which is not proportional to the horizon area and depend on the spin of the field considered and the characteristics of the black hole except that the different fields obey different statistics. The quantum entropy of any spin field is different from that of the scalar field ($s = 0$) which is given by

$$S_{\text{Scalar}} = \frac{A_H}{48\pi\epsilon^2} + \left[\frac{4\pi r_H}{45\beta} + \frac{\Lambda r_H^2}{90}\right] \ln\frac{\Gamma}{\epsilon}, \tag{20}$$

The disagreement exists even if we just set $s = 0$ in Eq. (19) (by an overall 2 factor for the Bose case and 7/8 for the Fermi case). This means that the whole expression of the statistical entropy of a spin field does not take the form of the scalar field. From Eq. (15), we learn that the contribution of the particles of the spin fields depends on its spin states.

On the other hand, when $\eta = 0$ and $\Lambda = 0$, so $\beta = 4\pi r_H = 8\pi M$, Eq. (20) give the result of the Schwarzschild black hole, which is equal to the Solodukhin's result(2), i.e., $S^{\text{Sch}}_{\text{Scalar}} = A_H/48\pi\epsilon^2 + (1/45)\ln(\Gamma/\epsilon)$. The first term in Eq. (20) is just the classical Bekenstein-Hawking entropy of a black hole(1).

In the entropy (19), the quadratically divergent term can be regarded as a renormalization of the gravitational constant determined by $1/G_{\text{ren}} = 1/G + \lambda\omega/12\pi\epsilon^2$ *(where* $\lambda = 1$ *for Bosonic field and* $\lambda = 7/2$ *for Fermionic field) as discussed in Refs. (8,9,25), the whole logarithmic divergence* $\zeta = \xi(M, s, etc.)\ln(\Gamma/\epsilon)$ *can be absorbed through defining the quantum-corrected radius of the horizon* $r^2_{H,\text{ren}} = r^2_H + \zeta l^2_{lp}$ *(where* $l^2_{lp} = G_{\text{ren}}$ *is Planck length).*

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